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# An Approximation Scheme for Gauss Curvature Flow

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**§1 Introduction.** We propose here an approximation scheme for Gauss curvature flow of a convex hypersurface in  $\mathbf{R}^{n+1}$  and explain how to prove the convergence of the scheme to the Gauss curvature flow.

The Gauss curvature flow of a convex hypersurface in  $\mathbf{R}^{n+1}$  is described as follows. Let  $\Gamma_0$  be a convex hypersurface of  $\mathbf{R}^{n+1}$  and  $F_0 : S^n \rightarrow \mathbf{R}^{n+1}$  be a parametric representation of  $\Gamma_0$ . The Gauss curvature flow of this hypersurface  $\Gamma_0$  is a collection  $F(\cdot, t) : S^n \rightarrow \mathbf{R}^{n+1}$  of closed hypersurfaces with parameter  $t \in [0, T)$  which is a solution of the initial value problem

$$(1.1) \quad \begin{cases} \frac{\partial F}{\partial t}(s, t) = -K^\beta(s, t)n(s, t) & (0 < t < T, s \in S^n) \\ F(s, 0) = F_0(s) & (s \in S^n), \end{cases}$$

where  $\beta > 0$  is a constant and  $K(s, t)$  and  $n(s, t)$  denote the Gauss curvature and the outward unit normal vector, respectively, at  $F(s, t)$  of the hypersurface  $F(\cdot, t)$ .

W. J. Firey [F] proposed problem (1.1) with  $\beta = 1$  as a mathematical model of the wearing process of stones on beach by waves and studied some basic properties of the solution  $F$  of this problem. Afterwards, K. Tso [T] and then B. Chow [C] studied problem (1.1) and established the following existence theorem.

**Theorem 1.** *If  $F_0$  represents a smooth, strictly convex hypersurface, then there exist a positive  $T > 0$  and a unique smooth solution  $F : S^n \times [0, T) \rightarrow \mathbf{R}^{n+1}$  of (1.1)*

such that  $F(\cdot, t)$  represents a strictly convex set  $\Gamma_t \equiv F(S^n, t) \subset \mathbf{R}^{n+1}$  for all  $0 < t < T$  and  $\Gamma_t$  converge to a point as  $t \nearrow T$ .

Another way of describing Gauss curvature flow is the so-called level-set approach, in which the evolving convex hypersurfaces  $\Gamma_t$  are regarded as the 0-level set of a function  $u$  defined on  $\mathbf{R}^{n+1} \times [0, \infty)$ . More precisely, the idea is explained as follows. Given a compact convex hypersurface  $\Gamma_0$ , we choose a function  $g \in BUC(\mathbf{R}^{n+1})$  so that

$$(1.2) \quad \Gamma_0 = \{z \in \mathbf{R}^{n+1} \mid g(z) = 0\} \quad \text{and} \quad \{z \in \mathbf{R}^{n+1} \mid g(z) \leq 0\} \text{ is convex,}$$

and consider the initial value problem

$$(1.3) \quad \begin{cases} \text{(i)} & u_t + G(Du, D^2u) = 0 \quad \text{in } \mathbf{R}^{n+1} \times (0, \infty), \\ \text{(ii)} & u(z, 0) = g(z) \quad (z \in \mathbf{R}^{n+1}). \end{cases}$$

Here the function  $G : (\mathbf{R}^{n+1} \setminus \{0\}) \times \mathcal{S}(n+1) \rightarrow \mathbf{R}$  is defined by

$$G(p, X) = -|p| \left\{ \det_+ \left( \frac{1}{|p|} (I - \bar{p} \otimes \bar{p}) X (I - \bar{p} \otimes \bar{p}) + \bar{p} \otimes \bar{p} \right) \right\}^\beta$$

where  $\bar{p} = p/|p|$  and

$$\det_+ A = \prod_{i=1}^{n+1} \max\{\lambda_i, 0\} \quad \text{for } A \in \mathcal{S}(n+1),$$

with  $\lambda_i$  ( $i = 1, \dots, n+1$ ) denoting the eigenvalues of  $A \in \mathcal{S}(n+1)$ . Now, the (generalized) Gauss curvature flow of  $\Gamma_0$  is defined as the collection  $\{\Gamma_t\}_{t \geq 0}$  of the closed subsets

$$(1.4) \quad \Gamma_t = \{z \in \mathbf{R}^{n+1} \mid u(z, t) = 0\} \subset \mathbf{R}^{n+1}.$$

One of main results in [IS] (see [IS, Theorems 1.8 and 1.9]) states that if  $g \in BUC(\mathbf{R}^{n+1})$ , then there is a unique viscosity solution  $u \in BUC(\mathbf{R}^{n+1} \times [0, \infty))$  of (1.3) and that the collection  $\{\Gamma_t\}_{t \geq 0}$  defined by (1.4) is independent of the choice of  $g$ . See [IS] as well for the correct definition of viscosity solution for (1.3), (i). This assertion is,

of course, a generalization of a well-known, similar observation due to Chen-Giga-Goto [CGG] and Evans-Spruck [ES] for mean curvature flow and alike.

An argument in [CEI] guarantees that under assumption (1.2), if we set

$$(1.5) \quad V_t = \{z \in \mathbf{R}^{n+1} \mid u(z, t) \leq 0\},$$

then  $V_t$  is a convex set and  $\Gamma_t = \partial V_t$  for all  $t \geq 0$ . We shall also call the collection  $\{V_t\}_{t \geq 0}$  the generalized *Gauss curvature flow* of the convex body  $V_0$ .

See [CEI] for a discussion on the consistency of this level-set approach and the parametric representation approach based on (1.1).

In what follows we discuss only on the generalized Gauss curvature flow defined via the level-set approach as above and hence suppress the word “generalized” in the argument below.

**§2 An approximation scheme and the main result.** Now, we introduce an approximation scheme for Gauss curvature flow. We need notation. We denote by  $\mathcal{C}(m)$  the collection of all closed subsets of  $\mathbf{R}^m$ . Let  $A \in \mathcal{C}(n+1)$  and  $p \in S^n$ . Define

$$(2.1) \quad \ell_0(A, p) = \sup\{\langle z, p \rangle \mid z \in A\}.$$

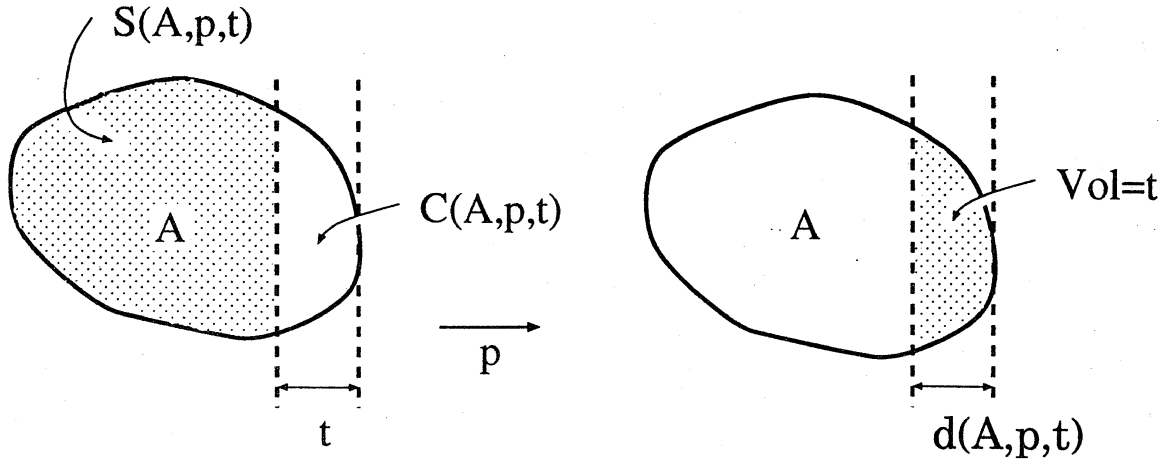
Of course,  $\ell_0(\emptyset, p) = -\infty$  and, if the set  $\{\langle z, p \rangle \mid z \in A\}$  is not bounded above,  $\ell_0(A, p) = \infty$ .

For  $t > 0$  define  $S(A, p, t)$  and  $C(A, p, t)$ , subsets of  $\mathbf{R}^{n+1}$ , by

$$(2.2) \quad S(A, p, t) = \{z \in A \mid \langle z, p \rangle \leq \ell_0(A, p) - t\}$$

and

$$(2.3) \quad C(A, p, t) = \{z \in A \mid \langle z, p \rangle > \ell_0(A, p) - t\} (= A \setminus S(A, p, t)).$$



Moreover, we define  $d(A, p, t) \in [0, \infty]$  by

$$(2.4) \quad d(A, p, t) = \inf \{s > 0 \mid \mathcal{L}^{n+1}(C(A, p, s)) \geq t\},$$

where  $\mathcal{L}^{n+1}(B)$  denotes the  $(n+1)$ -dimensional Lebesgue measure of the set  $B$ , and for  $\mu > 0$  set

$$(2.5) \quad d_\mu(A, p, t) = \min \{d(A, p, t), \mu\}$$

Finally, for any  $A \in \mathcal{C}(n+1)$ ,  $h > 0$ , and  $\mu > 0$  we define

$$(2.6) \quad T_h^\mu(A) = \bigcap_{p \in S^n} S \left( A, p, d_\mu \left( A, p, \alpha_n h^{\frac{1}{2\beta}} \right)^{\beta(n+2)} \right),$$

where

$$\alpha_n = \frac{2^{\frac{n+2}{2}} \omega_n}{n+2}, \quad \text{with } \omega_n = \text{the volume of the unit ball } \subset \mathbf{R}^n.$$

It is clear that for all  $A \in \mathcal{C}(n+1)$ ,  $T_h^\mu(A) \subset A$  and  $T_h^\mu(A) \in \mathcal{C}(n+1)$  and if  $A$  is convex then so is  $T_h^\mu(A)$ .

Fix a compact convex set  $V_0 \subset \mathbf{R}^{n+1}$ . Fix  $h > 0$  and  $\mu > 0$ . Define the sequence  $\{C_i\}_{i \in \mathbf{N}}$  of subsets of  $\mathbf{R}^{n+1}$  by the recursion formula

$$C_1 = V_0 \quad \text{and} \quad C_{i+1} = T_h^\mu(C_i) \quad \text{for } i \in \mathbf{N}$$

and the collection  $\{V_t^{\mu,h}\}_{t \geq 0}$  of subsets of  $\mathbf{R}^{n+1}$  by

$$(2.7) \quad V_t^{\mu,h} = C_i \quad \text{if } (i-1)h \leq t < ih \quad \text{and } i \in \mathbf{N}.$$

This collection  $\{V_t^{\mu,h}\}_{t \geq 0}$ , with  $\mu > 0$  and  $h > 0$ , is our approximation scheme for the Gauss curvature flow  $\{V_t\}_{t \geq 0}$  defined by (1.5).

The main result in this paper is the following

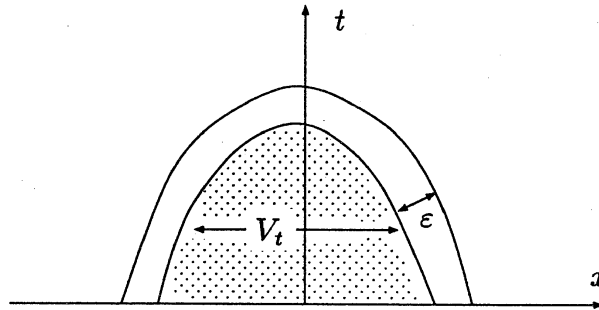
**Theorem 2.** *Assume that  $V_0$  is compact and convex,  $\beta \geq \frac{1}{n+2}$ , and  $\mu \in (0, \frac{1}{6})$ . For each  $\varepsilon > 0$  there is a  $\delta > 0$  such that if  $0 < h < \delta$ , then*

$$(2.8) \quad \bigcup_{t \geq 0} V_t \times \{t\} \subset \bigcup_{t \geq 0} V_t^{\mu,h} \times \{t\} + B^{n+2}(0, \varepsilon)$$

and

$$(2.9) \quad \bigcup_{t \geq 0} V_t^{\mu,h} \times \{t\} \subset \bigcup_{t \geq 0} V_t \times \{t\} + B^{n+2}(0, \varepsilon),$$

where  $B^{n+2}(0, \varepsilon)$  denotes the closed ball of radius  $\varepsilon$  and center at the origin in  $\mathbf{R}^{n+2}$ . That is, as  $h \searrow 0$ , the sets  $\bigcup_{t \geq 0} V_t^{\mu,h} \times \{t\}$  converge to the set  $\bigcup_{t \geq 0} V_t \times \{t\}$  in the Hausdorff distance.



The underlying idea of our definition of  $T_h^\mu$  (or our approximation scheme) can be explained as follows. Let  $A \in \mathcal{C}(n+1)$  be a smooth, strictly convex domain and  $B_0 \in \partial A$ .

Now, we assume that  $B_0 = 0$  and  $(0, -1) \in \mathbf{R}^n \times \mathbf{R}$  is the outward unit normal vector of  $A$  at 0. Writing  $z = (x, y) \in \mathbf{R}^n \times \mathbf{R}$  for generic points in  $\mathbf{R}^{n+1}$ , we may claim that in a neighborhood of 0, the set  $A$  is almost identical to the paraboloid

$$P = \{(x, y) \mid y \geq \frac{1}{2}(\kappa_1 x_1^2 + \cdots + \kappa_n x_n^2)\},$$

where  $\kappa_i$  denotes the principal curvatures of the surface  $\partial A$  at 0.

For  $d > 0$  we denote

$$P(d) = P \cap \{(x, y) \mid y \leq d\},$$

and compute the volume of the set  $P(d)$ , to find

$$(2.10) \quad \mathcal{L}^{n+1}(P(d)) = \alpha_n \frac{d^{\frac{n+2}{2}}}{\sqrt{K}},$$

$K$  denoting the Gauss curvature of  $\partial A$  at 0, i.e.,  $K = \prod_{i=1}^n \kappa_i$ .

If a convex body, starting with  $A$  at time 0, is shrinking with velocity

$$v = K^\beta,$$

in the directions of inward normal vectors of  $A$ , then the boundary point of the convex body at time  $h > 0$  with  $(0, -1)$  as its outward normal direction must be somewhere near the point  $B_1$  with coordinates

$$(0, K^\beta h).$$

If we set

$$d = (K^\beta h)^{\frac{1}{\beta(n+2)}},$$

and plug this into (2.10), then we find the formula

$$(2.11) \quad \mathcal{L}^{n+1}(P(d)) = \alpha_n h^{\frac{1}{2\beta}}.$$

A nice feature in formula (2.11) is that it does not involve the Gauss curvature  $K$  explicitly any more. Moreover, the formula determines  $d$  uniquely as a function of  $h > 0$ . Thus, reversing the above process, i.e., fixing first  $d > 0$  by formula (2.11) and then setting  $a = d^{\beta(n+2)}$ , we can identify the point  $B_1$  as the point with coordinates  $(0, a)$  without knowing the Gauss curvature  $K$ .

Roughly speaking, the set  $T_h^\mu(A)$  is defined as the convex hull of all the points  $B_1$  obtained from  $B_0 \in \partial A$  by the process described above.

**§3 Some properties of  $T_h^\mu$ .** We begin with the following proposition, which says that the mapping  $T_h^\mu$  is invariant under translation and orthogonal transformation. Henceforth  $h$ ,  $\beta$ , and  $\mu$  denote fixed positive constants.

**Proposition 1.** *For any  $A \in \mathcal{C}(n+1)$ ,  $U \in O(n+1)$ , and  $z \in \mathbf{R}^{n+1}$  we have*

$$(3.1) \quad T_h^\mu(U(A)) = U(T_h^\mu(A)),$$

$$(3.2) \quad T_h^\mu(z + A) = z + T_h^\mu(A).$$

Here and later  $O(n+1)$  denotes the set of orthogonal matrices of order  $n+1$ .

The proof is straightforward and left to the reader.

The next proposition asserts the monotonicity of  $T_h^\mu$ .

**Proposition 2.** *Assume that  $\beta \geq \frac{1}{n+2}$  and  $0 < \mu < \frac{1}{6}$ . Then, for any  $A, B \in \mathcal{C}(n+1)$ , if  $A \subset B$  we have*

$$(3.3) \quad T_h^\mu(A) \subset T_h^\mu(B).$$

*Remark.* The restriction that  $\beta \geq \frac{1}{n+2}$  and  $\mu < \frac{1}{6}$  in Theorem 2 is due to the above proposition. The condition,  $\mu < \frac{1}{6}$ , is not optimal in this respect and we will not seek for the optimal one here.

The following property will be needed in the proof of Theorem 2.

**Proposition 3.** *Assume that  $\beta \geq \frac{1}{n+2}$  and  $0 < \mu < \frac{1}{6}$ . Let  $A_\varepsilon \in \mathcal{C}(n+1)$ , with  $0 < \varepsilon \leq 1$ , be compact and satisfy*

$$A_\varepsilon \subset A_\delta \quad \text{if } \varepsilon < \delta.$$



Then we have

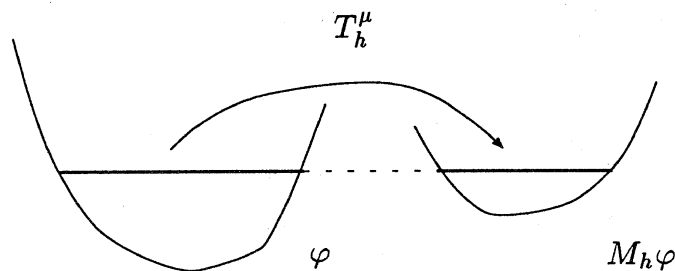
$$T_h^\mu\left(\bigcap_{0 < \varepsilon \leq 1} A_\varepsilon\right) = \bigcap_{0 < \varepsilon \leq 1} T_h^\mu(A_\varepsilon).$$

**§4 Level-set approach.** We shall take the level-set approach to proving Theorem 2 and here introduce the level-set approach to our approximation scheme.

In what follows we fix  $\beta \geq \frac{1}{n+2}$  and  $\mu \in (0, \frac{1}{6})$ . Fix  $h > 0$  as well. For any function  $\varphi \in C(\mathbf{R}^{n+1})$ , following [E], we set

$$(4.1) \quad M_h\varphi(z) = \inf\{\lambda \in \mathbf{R} \mid z \in T_h^\mu(\{\varphi \leq \lambda\})\} \quad (z \in \mathbf{R}^{n+1}).$$

Here and henceforth we use the notation  $\{P\}$  for  $\{z \mid P(z)\}$ , where  $P$  or  $P(z)$  is a proposition concerning  $z$ .



We see immediately from (4.1) that for  $\varphi \in C(\mathbf{R}^{n+1})$  and  $\lambda \in \mathbf{R}$ ,

$$T_h^\mu(\{\varphi \leq \lambda\}) \subset \{M_h\varphi \leq \lambda\} \quad \text{and} \quad \{M_h\varphi \leq \lambda\} \subset \bigcap_{\gamma > \lambda} T_h^\mu(\{\varphi \leq \gamma\}).$$

Loosely speaking, these say that  $T_h^\mu$  maps the sub-level set of  $\varphi$  of height  $\lambda$  to the sub-level set of  $M_h\varphi$  of height  $\lambda$ . In other words, the mapping  $T_h^\mu$  on sets of  $\mathbf{R}^{n+1}$  can be understood by seeing the mapping  $M_h$  on functions in  $\mathbf{R}^{n+1}$ .

Fix  $\varphi \in C(\mathbf{R}^{n+1})$ . Since  $T_h^\mu(\{\varphi \leq \lambda\}) \subset \{\varphi \leq \lambda\}$  for  $\lambda \in \mathbf{R}$ , we see that for all  $\varphi \in C(\mathbf{R}^{n+1})$ ,

$$(4.2) \quad M_h\varphi \geq \varphi \quad \text{in } \mathbf{R}^{n+1}.$$

Also, it follows from Proposition 2 that if  $\varphi, \psi \in C(\mathbf{R}^{n+1})$  and  $\varphi \leq \psi$  in  $\mathbf{R}^{n+1}$ , then

$$(4.3) \quad M_h \varphi \leq M_h \psi \quad \text{in } \mathbf{R}^{n+1}.$$

It follows that  $M_h \varphi$  is a real-valued function on  $\mathbf{R}^{n+1}$ .

Proposition 1 has direct consequences for  $M_h$ . Indeed, for any  $\varphi \in C(\mathbf{R}^{n+1})$  we have

$$(4.4) \quad (M_h \varphi) \circ U = M_h(\varphi \circ U) \quad \text{for all } U \in O(n+1),$$

where  $U \in O(n+1)$  is regarded as a mapping, and

$$(4.5) \quad M_h \circ \tau_y = \tau_y \circ M_h \quad \text{for all } y \in \mathbf{R}^{n+1},$$

where  $\tau_y$  denotes the translation by  $y$ , i.e.,  $\tau_y \varphi(z) = \varphi(z - y)$ . The proof of these claims are again left to the reader.

Next, we observe that  $M_h \varphi \in UC(\mathbf{R}^{n+1})$  for all  $\varphi \in UC(\mathbf{R}^{n+1})$ . This will be proved as a consequence of (4.3), (4.5), and the following claim.

Let  $\theta \in C(\mathbf{R})$  be any nondecreasing function. The claim is:

$$(4.6) \quad M_h(\theta \circ \varphi) = \theta \circ (M_h \varphi) \quad \text{for } \varphi \in C(\mathbf{R}^{n+1}).$$

The proof of this claim, which is again easy, is left to the reader. (See (2.10) in [I1] for a similar observation.)

To conclude the uniform continuity, let  $\varphi \in UC(\mathbf{R}^{n+1})$  and  $\omega$  denote the modulus of continuity of  $\varphi$ . If  $y \in \mathbf{R}^{n+1}$ , then

$$\tau_y \varphi \leq \varphi + \omega(|y|),$$

and so, using (4.5), (4.3), and (4.6) with  $\theta(r) = r + \omega(|y|)$ , we see that

$$M_h \varphi(z - y) = M_h(\tau_y \varphi)(z) \leq M_h \varphi(z) + \omega(|y|) \quad (z \in \mathbf{R}^{n+1}),$$

from which follows the uniform continuity of  $M_h\varphi$ , i.e.,

$$(4.7) \quad |M_h\varphi(z) - M_h\varphi(y)| \leq \omega(|z - y|) \quad \text{for all } z, y \in \mathbf{R}^{n+1}.$$

Similarly, we have

$$(4.8) \quad \|M_h\varphi - M_h\psi\| \leq \|\varphi - \psi\| \quad \text{for } \varphi, \psi \in C(\mathbf{R}^{n+1}),$$

where  $\|\varphi\| = \sup_{\mathbf{R}^{n+1}} |\varphi| \in [0, \infty]$ .

Also, we easily see that if  $c$  is a constant function on  $\mathbf{R}^{n+1}$  then

$$M_h c = c.$$

Our proof of Theorem 2 will be carried out via the following

**Theorem 3.** *Let  $g \in BUC(\mathbf{R}^{n+1})$  be such that for any  $\lambda < \sup_{\mathbf{R}^{n+1}} g$ , the set  $\{g \leq \lambda\}$  is compact and convex. Let  $u \in BUC(\mathbf{R}^{n+1} \times [0, \infty))$  be the viscosity solution of (1.3). Define  $v_h : \mathbf{R}^{n+1} \times [0, \infty) \rightarrow \mathbf{R}$  by*

$$(4.9) \quad v_h(z, t) = M_h^i g(z) \quad \text{if } (i-1)h \leq t < ih \text{ and } i \in \mathbf{N},$$

where  $M_h^i$  denotes the  $i$  times iterates of the mapping  $M_h$ . Then for each  $0 < T < \infty$ , as  $h \searrow 0$ ,

$$(4.10) \quad v_h(z, t) \rightarrow u(z, t) \quad \text{uniformly on } \mathbf{R}^{n+1} \times [0, T].$$

The above definition (4.9) is a reformulation of (2.7) in terms of the level-set approach. (See the next Proposition.)

Let us state here a corollary of Proposition 3, which gives a better connection between (4.9) and (2.7).

**Proposition 4.** *Let  $\gamma \in \mathbf{R}$  and  $\varphi \in C(\mathbf{R}^{n+1})$  be such that  $\{\varphi \leq \gamma\}$  is a compact set. Then, for  $z \in \mathbf{R}^{n+1}$ , if  $M_h\varphi(z) < \gamma$ , then we have*

$$(4.11) \quad M_h\varphi(z) = \min\{\lambda \in \mathbf{R} \mid z \in T_h^\mu(\{\varphi \leq \lambda\})\}.$$

Note that under the above hypothesis, if  $\lambda < \gamma$  then we have

$$\{M_h\varphi \leq \lambda\} \subset T_h^\mu(\{\varphi \leq \lambda\}).$$

*Proof.* Assume that  $z \in \mathbf{R}^{n+1}$  satisfies  $\lambda \equiv M_h\varphi(z) < \gamma$ .

It follows that if  $t > \lambda$  then

$$(4.12) \quad z \in M_h(\{\varphi \leq t\}).$$

Fix any  $\eta \in (\lambda, \gamma)$ . Note that

$$\{\varphi \leq \lambda\} = \bigcap_{\lambda < t \leq \eta} \{\varphi \leq t\}.$$

Now, from Proposition 3 and (4.12), we have

$$T_h^\mu(\{\varphi \leq \lambda\}) = \bigcap_{\lambda < t \leq \eta} T_h^\mu(\{\varphi \leq t\}) \ni z,$$

whence follows (4.11).  $\square$

**§5 Approximate derivative of  $M_h$  at  $h = 0$ .** In this section we assume that  $\beta \geq \frac{1}{n+2}$  and  $\mu \in (0, \frac{1}{6})$ .

The key observation in the proof of Theorem 3 will be stated in this section, which roughly says that the generator of Gauss curvature flow in terms of the level-set approach, i.e.,  $-G$  in (1.3), (i) “approximates the derivative” of  $M_h$  at  $h = 0$ .

Indeed, we have the following two theorems. The reader who is interested in the proof of these theorems should consult [I2].

**Theorem 4.** *Let  $\varphi \in C^2(\mathbf{R}^{n+1})$  satisfy  $D\varphi(\hat{z}) \neq 0$ , with  $\hat{z} \in \mathbf{R}^{n+1}$ . Then for each  $\varepsilon > 0$  there is a constant  $\delta > 0$  such that*

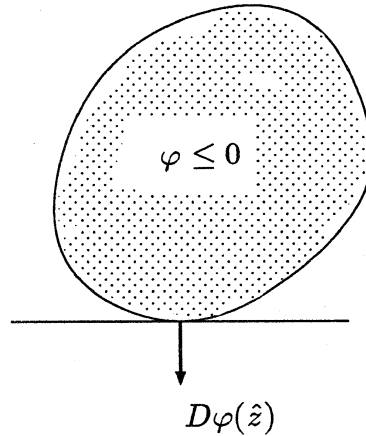
$$M_h\varphi(z) \leq \varphi(z) + (-G(D\varphi(\hat{z}), D^2\varphi(\hat{z})) + \varepsilon)h \quad (z \in B^{n+1}(\hat{z}, \delta), h \in (0, \delta)).$$

**Theorem 5.** Let  $\varphi \in C^2(\mathbf{R}^{n+1})$  satisfy  $D\varphi(\hat{z}) \neq 0$ , with  $\hat{z} \in \mathbf{R}^{n+1}$ . Assume that

$$\begin{cases} \varphi(z) > \varphi(\hat{z}) & \text{if } \langle z, D\varphi(\hat{z}) \rangle \geq 0 \text{ and } z \neq \hat{z}, \\ \liminf_{|z| \rightarrow \infty} \varphi(z) > \varphi(\hat{z}), \\ -G(D\varphi(\hat{z}), D^2\varphi(\hat{z})) > 0. \end{cases}$$

Then for each  $\varepsilon > 0$  there is a constant  $\delta > 0$  such that

$$M_h\varphi(z) \geq \varphi(z) + (-G(D\varphi(\hat{z}), D^2\varphi(\hat{z})) - \varepsilon)h \quad (0 < h \leq \delta, z \in B^{n+1}(\hat{z}, \delta)).$$



**§6 Convergence of the approximation scheme.** We shall complete the proof of Theorems 2 and 3 in this section.

We will again assume throughout that  $\beta \geq \frac{1}{n+2}$  and  $0 < \mu < \frac{1}{6}$ .

*Proof of Theorem 3.* Let  $g \in BUC(\mathbf{R}^{n+1})$ ,  $u \in BUC(\mathbf{R}^{n+1} \times [0, \infty))$ , and  $v_h : \mathbf{R}^{n+1} \times [0, \infty) \rightarrow \mathbf{R}$  be as in Theorem 3.

Define

$$\bar{v}(z, t) = \lim_{r \searrow 0} \sup \{v_h(y, s) \mid (y, s) \in \mathbf{R}^{n+1} \times [0, \infty) \cap B^{n+2}(z, t; r), 0 < h < r\}$$

$$((z, t) \in \mathbf{R}^{n+1} \times [0, \infty)),$$

and

$$\underline{v}(z, t) = \liminf_{r \searrow 0} \{v_h(y, s) \mid (y, s) \in \mathbf{R}^{n+1} \times [0, \infty) \cap B^{n+2}(z, t; r), 0 < h < r\} \\ ((z, t) \in \mathbf{R}^{n+1} \times [0, \infty)).$$

We shall first prove that  $u = \bar{v} = \underline{v}$  in  $\mathbf{R}^{n+1} \times [0, \infty)$ , which guarantees that  $v_h(z, t) \rightarrow u(z, t)$  uniformly on compact subsets of  $\mathbf{R}^{n+1} \times [0, \infty)$  as  $h \searrow 0$ .

To see this, we prove that  $\bar{v}$  and  $\underline{v}$  are a viscosity subsolution of (1.3), (i) and a viscosity supersolution of (1.3), (i), and that

$$\limsup_{r \searrow 0} \{|\bar{v}(x, t) - \underline{v}(y, s)| \mid |x - y| < r, 0 < t, s < r\} \leq 0.$$

Then we use a comparison theorem, to find that  $\bar{v} \leq \underline{v}$  in  $\mathbf{R}^{n+1} \times [0, \infty)$ . And we see that

$$u = \bar{v} = \underline{v} \quad \text{on } \mathbf{R}^{n+1} \times [0, \infty).$$

This shows that

$$v_h(x, t) \rightarrow u(x, t) \quad \text{as } h \searrow 0.$$

Here the convergence holds uniformly on any compact subsets of  $\mathbf{R}^{n+1} \times [0, \infty)$ .

A simple modification of the above arguments yields the uniform convergence in the whole space  $\mathbf{R}^{n+1} \times [0, \infty)$ .  $\square$

*Proof of Theorem 2.* Fix any compact convex  $V_0 \subset \mathbf{R}^{n+1}$  and choose  $g \in BUC(\mathbf{R}^{n+1})$  so that  $\{g \leq 0\} = V_0$  and so that for any  $\lambda < \sup_{\mathbf{R}^{n+1}} g$  the set  $\{g \leq \lambda\}$  is compact and convex. Let  $u \in BUC(\mathbf{R}^{n+1} \times [0, \infty))$  be the viscosity solution of (1.3) and  $v_h : \mathbf{R}^{n+1} \times [0, \infty) \rightarrow \mathbf{R}$  be defined by (4.9).

First we observe in view of Proposition 4 that

$$V_t^{\mu, h} = \{v_h(\cdot, t) \leq 0\} \quad (h > 0, t \geq 0).$$

Next, since  $G \leq 0$ , we infer that for each  $z \in \mathbf{R}^{n+1}$  the function  $u(z, t)$  of  $t \geq 0$  is nondecreasing. By comparison between  $u$  and the constant function  $\sup_{\mathbf{R}^{n+1}} g$ , we

see that  $u \leq \sup_{\mathbf{R}^{n+1}} g$  in  $\mathbf{R}^{n+1} \times [0, \infty)$ . By comparing each Gauss curvature flow  $\{u(\cdot, t) \leq \lambda\}_{t \geq 0}$  with the Gauss curvature flow of a ball containing the set  $\{g \leq \lambda\}$ , we then conclude that

$$\lim_{t \rightarrow \infty} u(z, t) = \sup_{\mathbf{R}^{n+1}} g$$

uniformly in  $\mathbf{R}^{n+1}$ .

Fix  $\gamma > 0$  so that  $\gamma < \sup_{\mathbf{R}^{n+1}} g$ , and then  $T > 0$  so that

$$u(z, t) > \gamma \quad (z \in \mathbf{R}^{n+1}, t \geq T).$$

Fix  $\varepsilon > 0$  and, in view of the compactness of the set  $\{u \leq \gamma\}$  and the continuity of  $u$ , choose  $\delta \in (0, \gamma)$  so that

$$\{u \leq \delta\} \subset \{u \leq 0\} + B^{n+2}(0, \varepsilon).$$

By Theorem 3, we can choose  $\eta > 0$  so that for all  $0 < h \leq \eta$ ,

$$(6.1) \quad |u(z, t) - v_h(z, t)| \leq \delta \quad (z \in \mathbf{R}^{n+1}, t \in [0, T]).$$

Then, if  $0 < h \leq \eta$ ,  $(z, t) \in \mathbf{R}^{n+1} \times [0, T]$ , and  $v_h(z, t) \leq 0$ , we have

$$(z, t) \in \{u \leq \delta\} \subset \{u \leq 0\} + B^{n+2}(0, \varepsilon).$$

If  $0 < h \leq \eta$  and  $z \in \mathbf{R}^{n+1}$ , then, since  $u(z, T) > \gamma > \delta$ , we see that  $v_h(z, T) > 0$ . Noting that for each  $z \in \mathbf{R}^{n+1}$  the function  $v_h(z, t)$  of  $t \geq 0$  is nondecreasing, we find that if  $0 < h \leq \eta$ , then  $v_h(z, t) > 0$  for all  $(z, t) \in \mathbf{R}^{n+1} \times [T, \infty)$ .

Thus we see that

$$\{v_h \leq 0\} \subset \{u \leq 0\} + B^{n+2}(0, \varepsilon),$$

proving (2.9).

The other inclusion is observed with a little more care. Indeed, we need to divide our considerations into two cases.

*Case 1:*  $\text{Int } V_0 = \emptyset$ . The extinction time in this case is zero (see [CEI]), i.e.,  $V_t = \emptyset$  for  $t > 0$ . Therefore, it is obvious that  $\{u \leq 0\} \subset \{v_h \leq 0\}$  for all  $h > 0$ .

*Case 2:*  $\text{Int } V_0 \neq \emptyset$ . We may now assume that  $g(x) < 0$  for all  $x \in \text{Int } V_0$ . The convexity of  $V_0$  guarantees (see [CEI]) that there does not happen “fattening” in the Gauss curvature flow  $\{V_t\}_{t \geq 0}$ , from which we can conclude that

$$\{u \leq 0\} = \overline{\bigcup_{s>0} \{u \leq -s\}}.$$

Fix  $\varepsilon > 0$ , and we see from the above that for some  $\delta > 0$ ,

$$\{u \leq 0\} \subset \{u \leq -\delta\} + B^{n+2}(0, \varepsilon).$$

We may assume that (6.1) holds with current  $\delta > 0$  and some  $\eta > 0$ . Now it is immediate to conclude that for all  $0 < h \leq \eta$ ,

$$\{u \leq 0\} \subset \{v_h \leq 0\} + B^{n+2}(0, \varepsilon),$$

which proves (2.8).  $\square$



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